Large-N QCD at Strong Transverse Gauge Coupling

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Strong Frontier 2009, 12-17 January 2009, PPISR, Bangalore

Summary

QCD is analysed with two light-front continuum dimensions and two transverse lattice dimensions. In the limit of large number of colours and strong transverse gauge coupling, dynamical contributions of light-front and transverse directions factorise, and the theory can be analytically solved in a closed form. Exact functional integration over the gauge field produces an effective action for the quark bilinear field, which can be extremised in different hadronic sectors.

For the mesons, an integral equation is obtained, which generalises the ’t Hooft equation by including the spin degrees of freedom. Spectral properties of the mesons are extracted from this equation, while light-front wavefunctions and form factors can be obtained by its numerical solution.

For baryons, the effective action is extremised in the presence of twisted boundary conditions. An integral equation describing the valence quark density is obtained, which again requires numerical solution.

These results should be a good starting point to model QCD observables that only weakly depend on transverse directions, e.g. deep inelastic scattering structure functions.

Space-time:

$R^2 \otimes Z^2$ in coordinate space, $R^2 \otimes T^2$ in momentum space.

Metric: $$g_{\alpha\beta} = \frac{1}{2}\{\gamma_\alpha, \gamma_\beta\} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Limits:

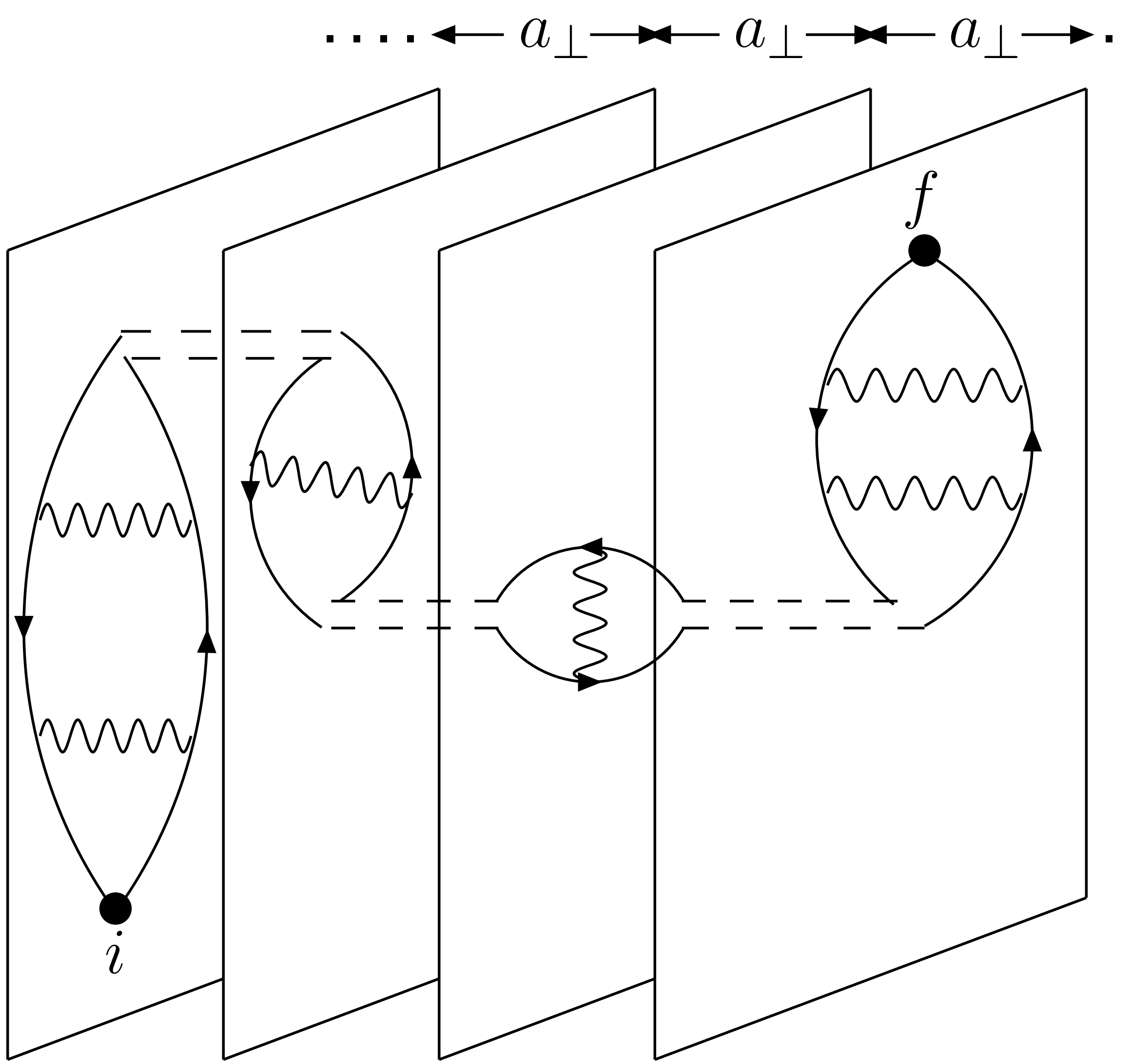
First $g_\perp \to \infty$, which makes $a_\perp = O(\Lambda_{QCD})$.

Then $N \to \infty$, yielding effective action at its stationary point.

Then $m \to 0$, producing spontaneous symmetry breaking in the chiral limit.

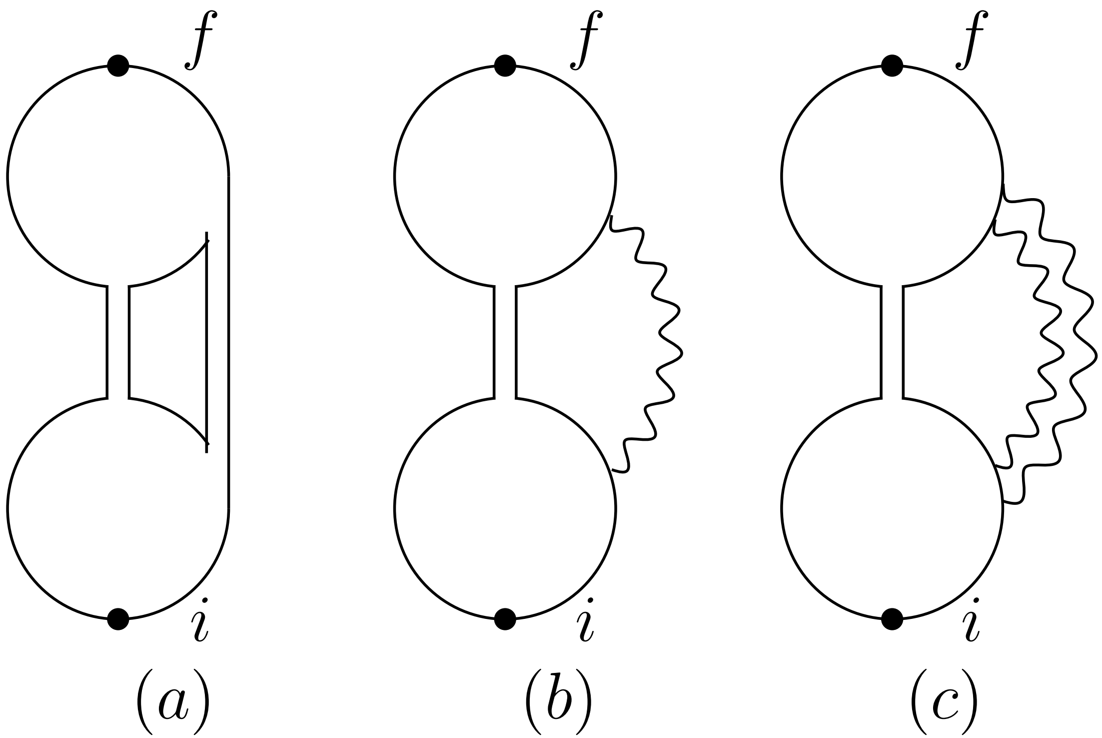
The $1/g_\perp$ expansion has a non-zero radius of convergence.

The $N \to \infty$ and $m \to 0$ limits do not commute.



A schematic representation of the meson propagator on a transverse lattice at large-N and strong coupling. The hyperplanes are coupled (no dimensional reduction). The colour singlet jumps between hyperplanes are shown by dashed lines.

Suppressed Contributions



$1/N$ corrections to the strong coupling transverse lattice large-N QCD:
(a) a quark loop is suppressed by N^{-1},
(b) a gluon jump between hyperplanes is suppressed by $N^{-1}g_{\perp}^{-2}$,
(c) a glueball jump is suppressed by N^{-2}.

$1/N$ corrections are hard to include, but $1/g_{\perp}$ corrections can in principle be included.

Features

The transverse lattice formulation can be interpreted as the result of an anisotropic Renormalisation Group evolution. (In statistical mechanics, an analogous technique is used to obtain high temperature series expansions.)

The theory for a single hyperplane is the 'tHooft model.

The hyperplanes do not decouple even for $g_\perp \to \infty$.
The strong coupling expansion is outside the physical region, nevertheless it has a finite radius of convergence.

Large-N limit leaves out sea quarks, still the remaining valence quarks contain enough non-perturbative physics.

Linear confinement is built into the formulation, due to linear Coulomb potential on the light-front, and due to completely disordered gauge field in the transverse directions.

The methodology can extend exact results of low dimension theories to the corresponding phases of higher dimensional theories.

Action

$$
\begin{aligned}
S &= a_\perp^2 \sum_{x_\perp} \int d^2x \left[-\frac{N}{4g^2} \sum_{\mu\nu a} F^a_{\mu\nu}(x) F^{\mu\nu a}(x) \right. \\
&+ \overline{\psi}(x) \Big(i \sum_\mu \gamma^\mu \partial_\mu - \sum_\mu \gamma^\mu A_\mu - m \Big) \psi(x) \\
&+ \frac{\kappa}{2a_\perp} \sum_n \Big\{ \overline{\psi}(x)(1 + i\gamma^n) U_n(x) \psi(x + \hat{n} a_\perp) \\
&+ \left. \overline{\psi}(x + \hat{n} a_\perp)(1 - i\gamma^n) U_n^\dagger(x) \psi(x) \Big\} \right] .
\end{aligned}
$$

Normalisation convention: g is held fixed as $N \to \infty$.
μ, ν label light-front directions, and n labels lattice directions.
Parameters $g, a_\perp, \kappa$ are to be non-perturbatively determined.

Dropped with $g_\perp \to \infty$: $\mathrm{Tr}[D_\mu U_n(x)(D^\mu U_n(x))^\dagger] = O(1/gg_\perp)$,
and $\mathrm{Tr}[U_n(x) U_m(x + \hat{n} a_\perp) U_n^\dagger(x + \hat{m} a_\perp) U_m^\dagger(x) = O(1/g_\perp^2)]$.

Functional Integration Procedure

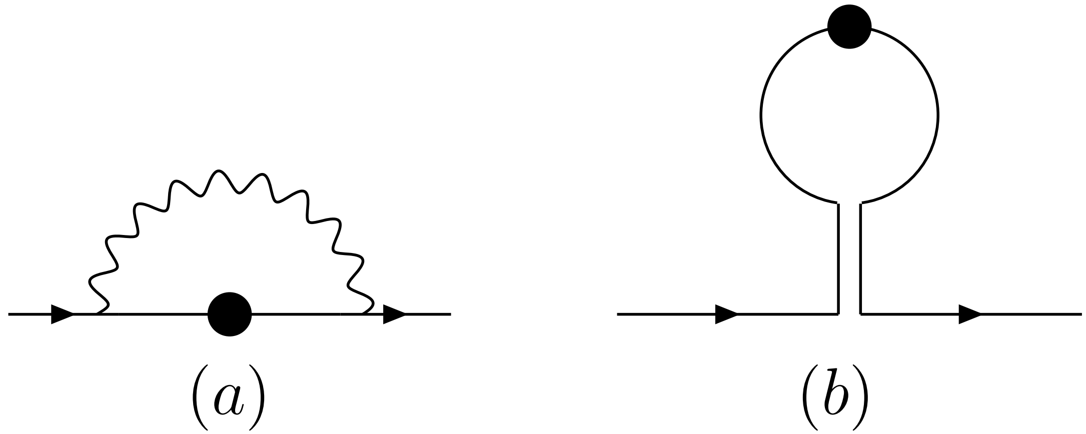
1. In the gauge $A^+ = 0$, functional integral over A^- is gaussian, which results in a linear confining potential along the light-front.

2. Action is linear in the links $U_\perp(x)$, whose integration in the $N \to \infty$ limit gives hopping term between adjacent light-fronts, for colour singlet local fermion bilinears.
   In case of $2+1$ dimensional Yang-Mills theory, the leading $1/g_\perp$ correction, $\mathrm{Tr}\left[D_\mu U_n(x)(D^\mu U_n(x))^\dagger\right]$, gives the principal chiral model. (V.P. Nair, P. Orland)

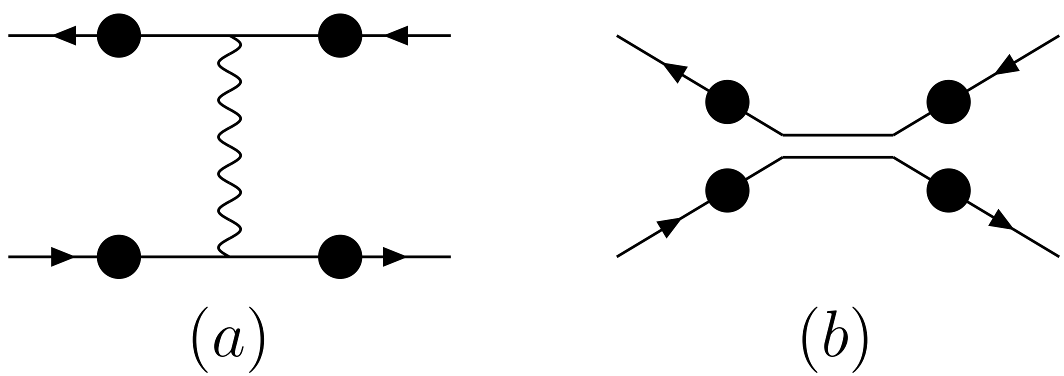
3. Non-local bosonisation using $\sigma_{\alpha\beta}(x,y) \equiv \overline{\psi}_\alpha(x)\psi_\beta(y)$ simplifies 4-Fermi interactions. Fermion bilinear terms in the effective action are then integrated exactly.

4. The effective action is explicitly proportional to N. So as $N \to \infty$, its stationary point gives the connected generating functional $W[\overline{\sigma}, J]$.

5. $\sigma(x,y)$ is gauge invariant for $x^- \neq y^-, x^+ = y^+, x_\perp = y_\perp$.



Self energy contributions to the quark propagator: (a) gluonic correction from the continuous dimensions, (b) tadpole correction due to hops in the lattice directions.



The interactions between quark and anti-quark in a meson: (a) gluon exchange in the continuous dimensions, (b) colour singlet hops in the lattice directions.

(Fermion line with a filled dot denotes the full quark propagator.)

Effective Action

Extremisation of the effective action yields (at $J = 0$):

$$
\begin{aligned}
1 &= i\overline{\sigma}^T(x,y)(i\partial\!\!\!/ - m)\delta^{(2)}(x-y) \\
&- \frac{ig^2}{2}|x^- - y^-|\,\overline{\sigma}^T(x,y)\gamma^+\overline{\sigma}^T(y,x)\gamma^+ \; - \; i\delta^{(2)}(x-y) \\
&\times \sum_n \Big[\overline{\sigma}^T(x,x)G(\overline{R})(1-i\gamma_n)\overline{\sigma}^T(x-n,x-n)(1+i\gamma_n) \\
&\qquad + \overline{\sigma}^T(x,x)(1+i\gamma_n)\overline{\sigma}^T(x+n,x+n)G(\overline{R})(1-i\gamma_n)\Big]
\end{aligned}
$$

Here $R = -(1-i\gamma_n)\sigma^T(x,x)(1+i\gamma_n)\sigma^T(x+n,x+n)$,
$G(R) = \frac{\kappa^2}{2(1+\sqrt{1+\kappa^2 R})}$, and units are chosen such that $a_\perp = 1$.

The meson stationary point is translationally invariant.
$\overline{\sigma}_{B=0}(x,x) \propto 1$ and $\overline{R}_{B=0} = 0$ make it independent of κ.

Fermion Bilinears

For Wilson fermions, the transverse tadpole insertions renormalising the quark mass vanish.

The chiral limit is at $m = 0$, as in the 't Hooft model.

The quark propagator along the light-front is

$$S(p) = \frac{i}{\not{p} - m - \Sigma(p) + i\epsilon} \, \delta_{x_\perp y_\perp} \ , \quad \Sigma(p) = \frac{-g^2 \gamma^+}{2\pi a_\perp^2 p^+} \ .$$

The chiral condensate is, as in the 't Hooft model [6],

$$\langle \overline{\psi} \psi \rangle_{(3+1)-\mathrm{dim}} \ = \ \frac{2}{a_\perp^2} \langle \overline{\psi} \psi \rangle_{(1+1)-\mathrm{dim}} \ \stackrel{m \to 0}{\longrightarrow} \ - \frac{N}{a_\perp^3} \sqrt{\frac{g^2}{3\pi}} \ .$$

The Goldstone boson decay constant is

$$(f_\pi)_{(3+1)-\mathrm{dim}} \ = \ \sqrt{2} (f_\pi)_{(1+1)-\mathrm{dim}} \ \stackrel{m \to 0}{\longrightarrow} \ \frac{1}{a_\perp} \sqrt{\frac{2N}{\pi}} \ .$$

$N = 3$, $f_\pi \simeq 132$ MeV $\Longrightarrow$ Lattice cut-off $(\pi/a_\perp) \simeq 300$ MeV.

Slope of π-trajectory $\Longrightarrow g^2/4\pi \simeq 2.3$, $\langle \overline{\psi} \psi \rangle \simeq -(165\mathrm{MeV})^3$.

Light-Front Meson States

The meson wavefunction with spin-parity structure Γ is

$$\phi_\Gamma(p,q) = \langle\overline{\psi}(p-q)\Gamma\psi(q)|\mathrm{Meson}_\Gamma(p)\rangle\ .$$

Work in the reference frame where all external transverse momenta vanish (possible for 2- and 3-point functions).

Scale momentum to $p = (p^+ = 1, p^- = M^2/2, p_\perp = 0)$. Interactions are independent of "$-$" and "$\perp$" components.

Meson wavefunction projected on the light-front is

$$\Phi_\Gamma(q^+ \equiv x) = \int dq^- \ \frac{d^2 q_\perp}{(2\pi)^2}\ \phi_\Gamma(p,q)\ .$$

δ-function constraint in transverse directions makes orbital angular momentum L_z vanish. Allowed spin-parity quantum numbers for mesons are $J^P = 0^\pm, 1^\pm$.

Meson Integral Equation

For quark-antiquark pair masses $m_{1,2}$ and $\beta \equiv g^2/\pi a_\perp^2$,

$$\mu^2(x)\Phi(x) \equiv \left[M^2 - \frac{m_1^2-\beta}{x} - \frac{m_2^2-\beta}{1-x}\right]\Phi(x)$$

$$= \quad \frac{1}{2x(1-x)}\left[\frac{m_1^2}{2x}\gamma^+ + x\gamma^- + m_1\right]$$

$$\times \quad \int_0^1 \frac{dy}{2\pi}\Big\{ -\frac{g^2}{a_\perp^2}\mathrm{P}\frac{1}{(x-y)^2}\gamma^+\Phi(y)\gamma^+$$

$$+\kappa^2\left[2\Phi(y) + \sum_n \gamma^n\Phi(y)\gamma^n\right]\Big\}$$

$$\times \quad \left[\frac{\mu^2(x)}{2}\gamma^+ + \frac{m_2^2}{2(1-x)}\gamma^+ + (1-x)\gamma^- - m_2\right].$$

$\kappa \to 0$ limit smoothly reduces to the 't Hooft equation.

Meson Properties

Transverse lattice dynamics produces a "wavefunction at the origin" interaction term—independent of the longitudinal momentum fraction x, but dependent on the spin-parity.

In terms of continuum and lattice Clifford bases, the 16-component meson wavefunction is $\Phi = \sum_{C;L} \Phi_{C;L} \Gamma^{C;L}$.

$$\Gamma_C \otimes \Gamma_L = \left\{1, \gamma^+, \gamma^-, \tfrac{1}{2}[\gamma^+, \gamma^-]\right\} \otimes \left\{1, \gamma^1, \gamma^2, \tfrac{1}{2}[\gamma^1, \gamma^2]\right\}$$

In this basis, $\sum_n \gamma^n \Phi_{C;L} \gamma^n \propto \Phi_{C;L}$, and the meson integral equation block-diagnoalises in to four sets (labeled by "L") of four components each.

Parity is exact in each set. $L = n_1, n_2$ sets are degenerate.

In conventional notation, $\{\pi, a_1(0)\} \in \Phi_{C;n_1 n_2}$, the degenerate pair $\{\rho(n), a_1(n)\} \in \Phi_{C;n}$, and $\{\rho(0), \sigma\} \in \Phi_{C;1}$.

(…contd.)

The non-singular spin-parity dependent transverse interaction provides a zero-point energy shift, leading to parallel infinite towers of meson states in $M^2 - n$ plane.

Parity is not manifest, but can be related to exchange symmetry. Meson states in each tower alternate in parity, with the lowest state having negative parity.

π-tower is the lowest, and σ-tower is the highest.
g^2 controls the slope of the towers, and κ their separation.
Numerical fits are required to fix their optimal values.

For $m = 0$, $\Phi^{(n=1)}_{-;n_1 n_2} = 1$ is the massless pseudoscalar Goldstone boson of the theory.

For highly excited states, the interaction potential is irrelevant, yielding the “particle in a box" behaviour:

$$n \gg 1 : \quad \Phi^{(n)}_{-;n_1 n_2} \simeq \sqrt{2}\sin(n\pi x), \quad M_n^2 \simeq n\pi g^2/a_\perp^2 .$$

Baryon States

Baryons are semi-classical solitons in the $N \to \infty$ limit.
In the Hartree (mean field) approximation, the ground state baryons have all the valence quarks are in the same lowest state of the common static potential. The total wavefunction is fully symmetric in space with $I = J$.

The fixed baryon number constraint at every instant

$$\delta(Q(x^+) - NB) \; , \;\; Q(x^+) = a_\perp^2 \sum_{x_\perp} \int dx^- \; \overline{\psi}\gamma^+\psi(x, x_\perp),$$

can be incorporated using a chemical potential

$$\int [D\mu] \; \exp\left[i(Q - NB)\mu\right] \Longrightarrow \Sigma_{\mu\neq 0}(p) = -\left(\frac{g^2}{2\pi a_\perp^2 p^+} + \mu\right)\gamma^+.$$

The baryon stationary point of the effective action is not translationally invariant and obeys

$$a_\perp^2 \sum_{x_\perp} \int dx^- \; \mathrm{tr}\left(\overline{\sigma}_{\alpha\beta}(x, x)\gamma^+_{\alpha\beta}\right) = N.$$

(. . . contd.)

We use the ansatz (with normalisation $\int dx^-|f(x^-)|^2 = \frac{1}{4}$):

$$\overline{\sigma}_{B=1}(x,y) = \overline{\sigma}_{B=0}(x,y) + \delta_{x_\perp,y_\perp} f^*(x^-)\gamma^+ f(y^-)$$

The stationary point condition then becomes
(for the γ^+-component and in momentum space):

$$\int \frac{dk^+}{4\pi^2 k^+}|\tilde{f}(k^+)|^2 \quad + \quad \int \frac{dp^+}{2\pi}\frac{dq^+}{2\pi}\frac{dk^+}{2\pi}\mathrm{P}\left[\frac{1}{(p^+)^2}\right] \times \quad \tilde{f}^*(q^+)\tilde{f}(k^+)\tilde{f}^*(k^+ + p^+)\tilde{f}(q^+ + p^+) = 0$$

The transverse lattice dynamics contributes only through renormalisation of the quark mass, $\widetilde{m} = m + 8G(0)\langle\overline{\psi}\psi\rangle/N$.

The baryon valence quark density, $4|\tilde{f}(k^+)|^2$ with $k^+ \geq 0$, requires numerical solution of this integral equation. (Qualitative behaviour is as expected, a peak at $x = 1/N$.)

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