Understanding the Born Rule in Weak Measurements

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N. Gisin, Phys. Rev. Lett. 52 (1984) 1657
A. Patel and P. Kumar, Phys Rev. A (to appear), arXiv:1509.08253
S. Kundu, T. Roy, R. Vijayaraghavan, P. Kumar and A. Patel (in progress)

Abstract

Projective measurement is used as a fundamental axiom in quantum mechanics, even though it is discontinuous and cannot predict which measured operator eigenstate will be observed in which experimental run. The probabilistic Born rule gives it an ensemble interpretation, predicting proportions of various outcomes over many experimental runs. Understanding gradual weak measurements requires replacing this scenario with a dynamical evolution equation for the collapse of the quantum state in individual experimental runs. We revisit the framework to model quantum measurement as a continuous nonlinear stochastic process. It combines attraction towards the measured operator eigenstates with white noise, and for a specific ratio of the two reproduces the Born rule. This fluctuation-dissipation relation implies that the quantum state collapse involves the system-apparatus interaction only, and the Born rule is a consequence of the noise contributed by the apparatus. The ensemble of the quantum trajectories is predicted by the stochastic process in terms of a single evolution parameter, and matches well with the weak measurement results for superconducting transmon qubits.

Axioms of Quantum Dynamics

(1) Unitary evolution (Schrödinger):

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle \ , \quad i\frac{d}{dt}\rho = [H,\rho] \ .$$

Continuous, Reversible, Deterministic.

Pure state evolves to pure state.

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(2) Projective measurement (von Neumann):

$$|\psi\rangle \longrightarrow P_i|\psi\rangle/|P_i|\psi\rangle|,\ P_i = P_i^\dagger,\ P_iP_j = P_i\delta_{ij},\ \sum_i P_i = I.$$

Discontinuous, Irreversible, Probabilistic choice of "i".

Pure state evolves to pure state. Consistent on repetition.

$\{P_i\}$ is fixed by the measurement apparatus eigenstates. But there is no prediction for which "i" will occur in a particular experimental run.

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Instead, with Born rule and ensemble interpretation,

$$prob(i) = \langle\psi|P_i|\psi\rangle = Tr(P_i\rho) \ , \quad \rho \longrightarrow \sum_i P_i\rho P_i \ .$$

Pure state evolves to mixed state. Predicted expectation values are averages over many experimental runs.

Weak Measurements

Information about the measured observable is extracted from the system at a slow rate (e.g. by weak coupling). Stretching out the time scale can allow one to monitor collapse of the system to a measurement eigenstate.

Note: A measurement interaction is the one where the apparatus does not, for whatever reasons, remain in a superposition of pointer states.

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The answers are important for increasing accuracy of quantum control and feedback. Knowledge of what happens in a particular experimental run (and not just the ensemble average) can improve efficiency and stability.

The projective measurement axiom needs to be replaced by a different continuous stochastic dynamics.

Beyond Quantum Mechanics

Physical:

(1) Hidden variables with novel dynamics may produce quantum mechanics as an effective theory, e.g. the GRW spontaneous collapse mechanism.

(2) Ignored (but known) interactions can produce effects that modify quantum dynamics at macroscopic scales, e.g. effects of CMBR or gravity.

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Bypass:

Many worlds interpretation—each evolutionary branch is a different world, and we only observe the measurement outcome corresponding to the world we live in (anthropic principle).

None of these have progressed to the level where they can be connected to verifiable experimental consequences.

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Such a dynamical process exists! Gisin (1984)

Salient Features

A precise ratio of evolution towards the measurement eigenstates and unbiased white noise is needed to reproduce the Born rule as a constant of evolution.

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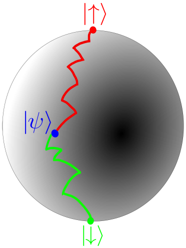
Technological advances allow us to monitor the quantum evolution during weak measurements. That can test the validity of the stochastic measurement formalism, and then help us figure out what may lie beyond.



Measurement $\equiv$ An effective process of a more fundamental theory.

Quantum Geodesic Trajectory

Leave out $i[\rho, H]$ from the evolution description for simplicity.
Unitary interpolation between ρ and P_i gives the geodesic evolution:

$$\boxed{\frac{d}{dt}\rho = g[\rho P_i + P_i\rho - 2\rho \; Tr(P_i\rho)] \; .}$$

g is the system-apparatus coupling, and t is the "measurement time".

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- For pure states, the equation can be written as:

$$\frac{d}{dt}\rho = -2g\mathcal{L}[\rho]P_i$$

This structure (involving the Lindblad operator) hints at an action-reaction relation between the dynamics of the system and the apparatus.

Ensemble of Quantum Geodesic Trajectories

The pointer basis $\{P_i\}$ is fixed by the system-apparatus interaction.
A criterion is needed to determine which of the many fixed points P_i will be approached in a particular experimental run.

Assign time-dependent real weights $w_i(t)$ to the evolution trajectory for P_i.

$$\frac{d}{dt}\rho = \sum_i w_i \; g[\rho P_i + P_i\rho - 2\rho Tr(P_i\rho)] \;, \quad \sum_i w_i = 1 \;.$$

Evolution still preserves $\rho^2 = \rho$. Every $\rho = P_i$ becomes a fixed point.
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The weighted trajectory evolution is:

$$\frac{d}{dt}(P_j\rho P_k) = P_j\rho P_k \; g[w_j + w_k - 2\sum_i w_i Tr(P_i\rho)] \; .$$

Diagonal projections of ρ fully determine the evolution:

$$\frac{2}{P_j\rho P_k}\frac{d}{dt}(P_j\rho P_k) = \frac{1}{P_j\rho P_j}\frac{d}{dt}(P_j\rho P_j) + \frac{1}{P_k\rho P_k}\frac{d}{dt}(P_k\rho P_k)$$

The evolution is totally decoupled from the decoherence process.

There are $n-1$ independent variables (diagonal projections $Tr(P_i\rho)$).

Choice of Trajectory Weights

The diagonal projections evolve according to:

$$\frac{d}{dt}d_j = 2g\ d_j(w_j - w_{\rm av})\ , \quad w_{\rm av} \equiv \sum_i w_i d_i\ .$$

Diagonal elements with $w_j > w_{\rm av}$ grow; those with $w_j < w_{\rm av}$ decay.

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Naive guess (instantaneous Born rule): $w_j = w_j^{IB} \equiv Tr(\rho(t) P_j)$

The evolution converges towards the subspace specified by the dominant diagonal projections of $\rho(t=0)$, i.e. the closest fixed points.

Though this result is consistent on repetition, it conflicts with experiments, because it is (i) deterministic and (ii) does not obey the Born rule.

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A way out: Instead of heading towards the nearest fixed point, the trajectories can be made to wander around the state space and explore other fixed points, by adding noise to the geodesic dynamics.
Properties of such a noise have to be found, while retaining $\sum_i w_i = 1$.

The type of the noise is not universal. It depends on the choice of the apparatus.

Quantum Diffusion: Single Qubit Measurement

The evolution equations simplify considerably for a qubit.

Let $|0\rangle$ and $|1\rangle$ be the measurement eigenstates.

$$\frac{d}{dt}\rho_{00} = 2g\ (w_0 - w_1)\rho_{00}\rho_{11}\ ,$$

$$\rho_{01}(t) = \rho_{01}(0)\left[\frac{\rho_{00}(t)\rho_{11}(t)}{\rho_{00}(0)\rho_{11}(0)}\right]^{1/2}\ .$$

With $\rho_{11}(t) = 1 - \rho_{00}(t)$ and $w_1(t) = 1 - w_0(t)$, only one independent variable describes evolution of the system.

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Evolution obeys Langevin dynamics, when unbiased white noise with spectral density S_ξ is added to w_i^{IB}. The trajectory weights become:

$$w_0 - w_1 = \rho_{00} - \rho_{11} + \sqrt{S_\xi}\ \xi\ .$$

$$\langle\langle \xi(t) \rangle\rangle = 0\ , \quad \langle\langle \xi(t)\xi(t') \rangle\rangle = \delta(t - t')\ .$$

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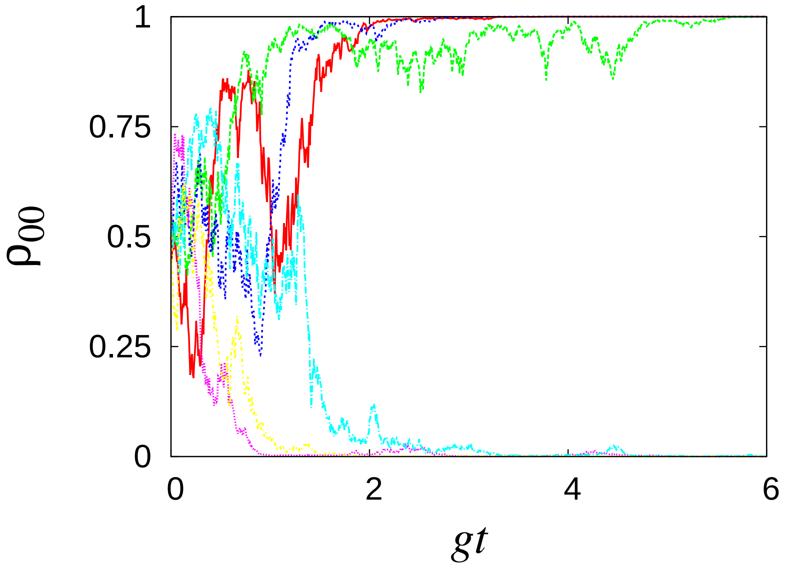
This is a stochastic differential process on the interval $[0, 1]$.
The fixed points at $\rho_{00} = 0, 1$ are perfectly absorbing boundaries.
A quantum trajectory would zig-zag through the interval before ending at one of the two boundary points.



Individual quantum evolution trajectories for the initial state $\rho_{00} = 0.5$, with measurement eigenstates $\rho_{00} = 0, 1$, and in presence of measurement noise satisfying $gS_\xi = 1$.

Single Qubit Measurement (contd.)

Let $P(x)$ be the probability that the initial state with $\rho_{00} = x$ evolves to the fixed point at $\rho_{00} = 1$. Then by symmetry,

$$P(0) = 0, \ P(0.5) = 0.5, \ P(1) = 1 \ .$$

No noise : $S_\xi = 0 \implies P(x) = \theta(x - 0.5)$.

Only noise : $S_\xi \to \infty \implies P(x) = 0.5$.

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It is instructive to convert the stochastic evolution equation from the differential Stratonovich form to the Itô form that specifies forward evolutionary increments:

$$d\rho_{00} = 2g\ \rho_{00}\rho_{11}(\rho_{00} - \rho_{11})(1 - gS_\xi)dt + 2g\sqrt{S_\xi}\ \rho_{00}\rho_{11}\ dW\ ,$$

$$\langle\langle dW(t)\rangle\rangle = 0\ ,\quad \langle\langle (dW(t))^2\rangle\rangle = dt\ .$$

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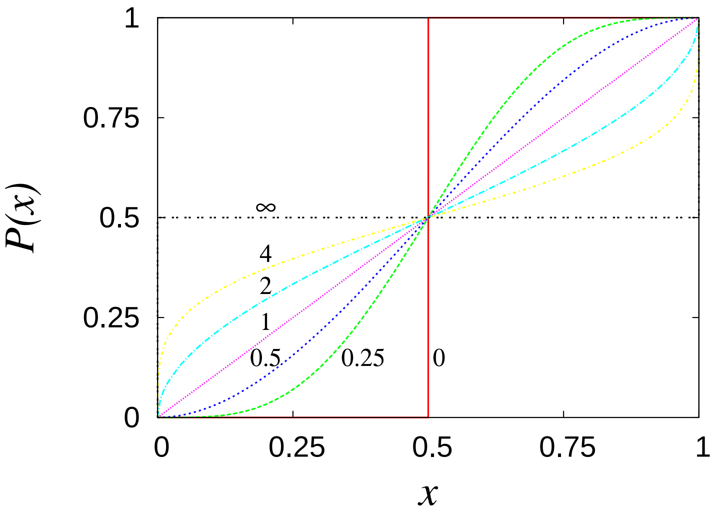
The first term produces drift in the evolution, while the second gives rise to diffusion. The evolution with no drift, i.e. the pure Wiener process with $gS_\xi = 1$, is rather special:

$$\langle\langle d\rho_{00} \rangle\rangle = 0 \iff \text{Born rule is a constant of evolution.}$$

Numerical tests were performed for different values of gS_ξ.



Probability that the initial qubit state $\rho_{00} = x$ evolves to the measurement eigenstate $\rho_{00} = 1$, for different values of the measurement noise. The gS_ξ values label the curves.

Ensemble Evolution Dynamics

During measurement, the probability distribution $p(\rho_{00}, t)$ of the set of quantum trajectories evolves according to the Fokker-Planck equation:

$$\frac{\partial p(\rho_{00},t)}{\partial t} = 2g\frac{\partial^2}{\partial^2\rho_{00}}\left(\rho_{00}^2(1-\rho_{00})^2 p(\rho_{00},t)\right) \ , \quad \text{with } gS_\xi = 1 \ .$$

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Its exact solution corresponding to initial $p(\rho_{00}, 0) = \delta(x)$ has two non-interfering components with areas x and $1-x$, monotonically travelling to the boundaries at $\rho_{00} = 1$ and 0 respectively.

Let $\tanh(z) = \rho_{00} - \rho_{11}$ map $\rho_{00} \in [0,1]$ to $z \in (-\infty, \infty)$. Then the two components are Gaussians centred at $z_\pm = z_0 \pm gt$, $z_0 = \tanh^{-1}(2x-1)$:

$$p(z,t) = \frac{1}{\sqrt{2\pi gt}} \left(x \exp\left[-\frac{(z-z_+)^2}{2gt} \right] + (1-x) \exp\left[-\frac{(z-z_-)^2}{2gt} \right] \right) .$$

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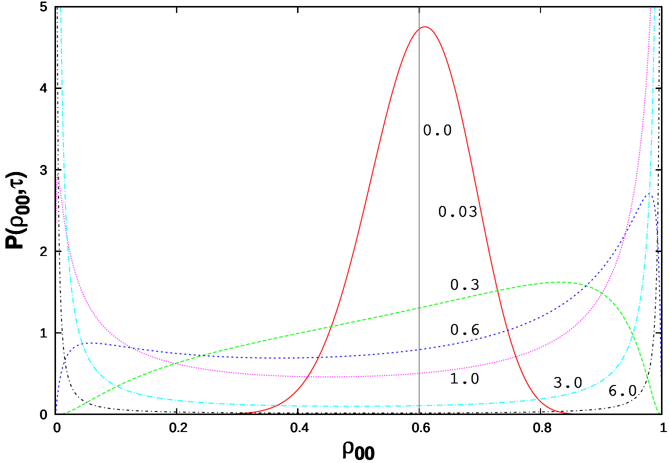
Parametric freedom: With the Born rule as a constant of evolution, g can be time-dependent, and gt is replaced by $\int_0^t g(t')dt'$.
The white noise distribution remains unspecified beyond the mean and the variance. Suitable choice can be made, e.g. Gaussian noise or Z_2 noise.



Distribution of the quantum measurement trajectories for quantum diffusion evolution of a qubit. The initial state is $\rho_{00}(\tau = 0) = 0.6$, and the curves are labeled by the values of the evolution parameter $\tau \equiv \int_0^t g(t')dt'$. The narrow initial distribution splits into two non-interfering components that converge to the measurement eigenstates at $\rho_{00} = 1, 0$ as $\tau \to \infty$.

For $\tau > 10$, 99% of the probability is within 1% of the two fixed points.

Experimental Setup

The system is a superconducting 3D transmon qubit.

Nonlinear oscillator consisting of a Josephson junction/SQUID shunted by a capacitor.

It possesses good coherence and is insensitive to charge noise.

Decoherence time $\sim 100\mu s$. Individual operation time: fraction of μs.

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The cavity frequency depends on the qubit state, whether $|0\rangle$ or $|1\rangle$.

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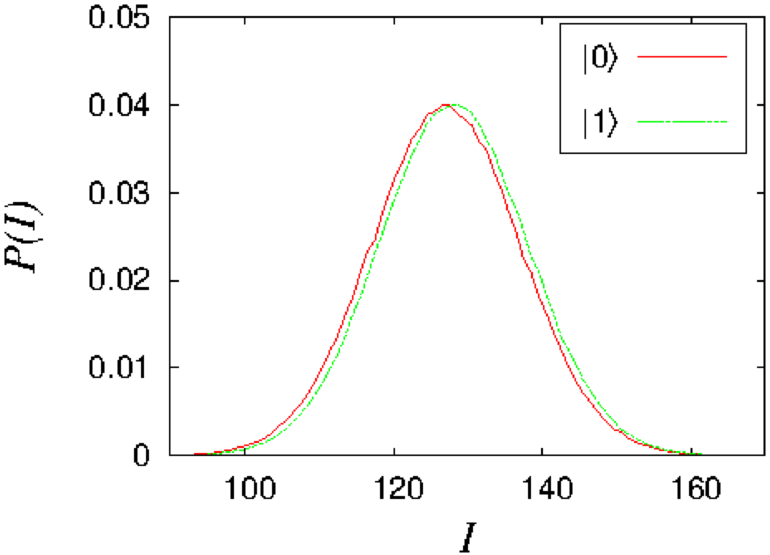
With a phase-sensitive amplifier, the scattering phase-shifts are Gaussians peaked at the two eigenvalues. Weak measurements result when the probe magnitude is small, making the two Gaussians closely overlap.



Observed probability distributions of the weak measurement Stern-Gerlach signal for the two eigenstates of a superconducting transmon qubit, for $\Delta t = 0.5\mu s$. They are approximate Gaussians, slightly displaced from each other ($\Delta I = 1.016 \ll \sigma = 9.99$).

Experimental Results

A quantum state initially polarised along X-axis is measured in the Z-basis.

The quantum state is infered from the integrated signal measurement, according to the Bayesian formalism (I_0, I_1, σ are known):

$$\frac{\rho_{00}(t)}{\rho_{11}(t)} = \frac{\rho_{00}(0)}{\rho_{11}(0)} \frac{\exp[-(I_m(t)-I_0)^2/2\sigma^2]}{\exp[-(I_m(t)-I_1)^2/2\sigma^2]} \ , \quad I_m(t) = \frac{1}{t}\int_0^t I(t')\ dt' \ .$$

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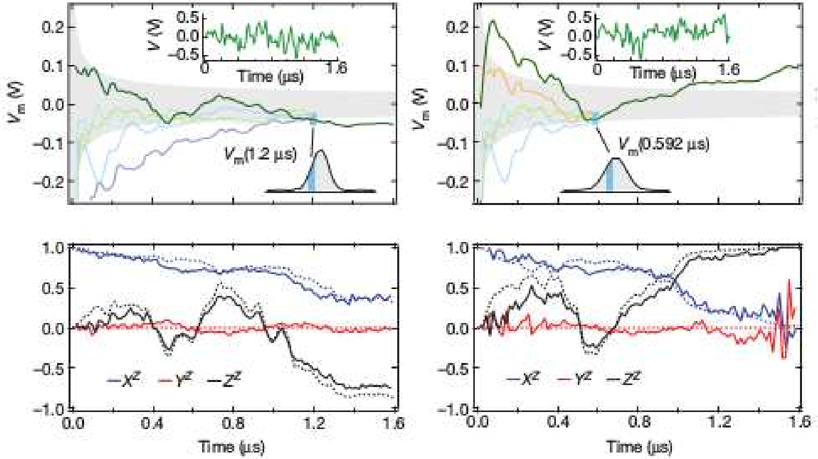
The experimentally observed trajectory distribution fits the quantum diffusion prediction very well, in terms of the single dimensionless evolution parameter $\tau \equiv \overline{g}t = \int_0^t g(t')dt'$:

$\chi^2 <$ few hundred, for 100 data points and one parameter.

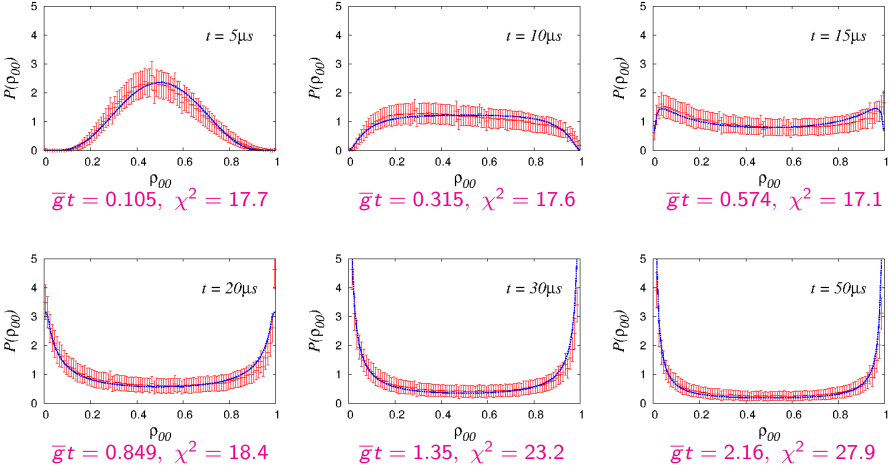
$\overline{g}t$ is almost linear in t, with a slower initial build-up.

Systematic errors: Initial state uncertainty, Excited state relaxation, Uncertainties in I_0, I_1, Higher excited state contamination. (Detector inefficiency can be absorbed in the value of $g(t)$.)



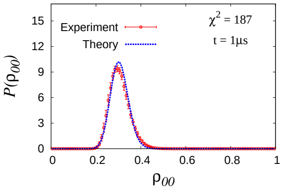
Observed quantum trajectories for weak Z-measurement of a superconducting qubit. The initial state is polarised along the X-axis. The top panels show the measured voltage distribution as a function of time, together with a few individual contributions. The lower panels display quantum trajectories obtained from the measured signal (dotted lines), and those reconstructed using tomography (solid lines).

Murch et al. (2013)

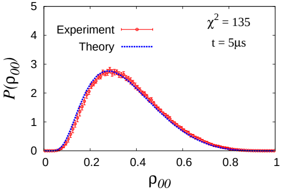


Time integrated coupling: $\bar{g}t = 4.7 \times 10^4 t - 0.1$

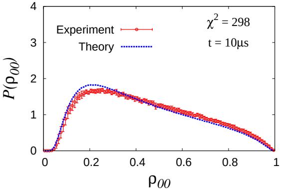
Evolution of the quantum trajectory distribution for weak Z-measurement of a superconducting transmon qubit initially polarised along the X-axis. The histograms represent the experimental data for an ensemble of 4×10^5 trajectories. The curves are the fits to the quantum diffusion distribution, with the single dimensionless evolution parameter $\tau \equiv \bar{g}t \in [0, 2.2]$.



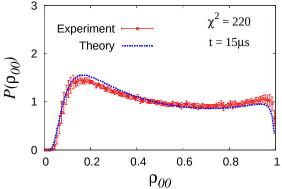
$\bar{g}t = 0.011$



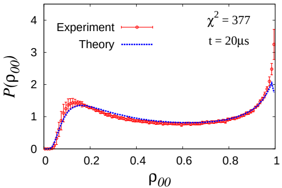
$\bar{g}t = 0.129$



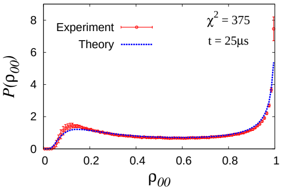
$\bar{g}t = 0.368$



$\bar{g}t = 0.646$

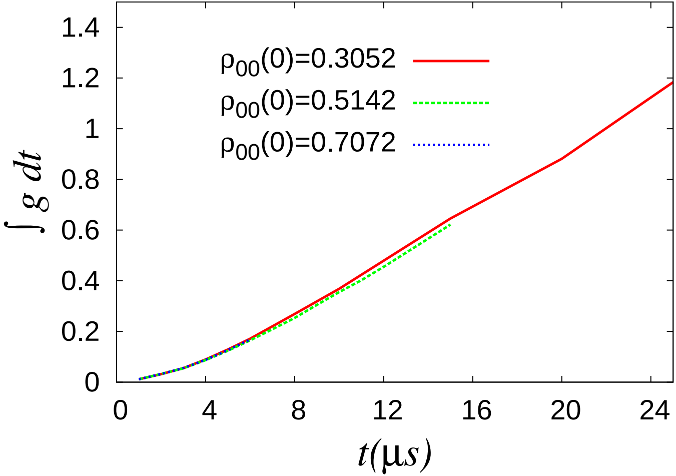


$\bar{g}t = 0.881$



$\bar{g}t = 1.183$

Evolution of the quantum trajectory distribution for weak Z-measurement of a superconducting transmon qubit with the initial state $\rho_{00} = 0.3$. The histograms represent the experimental data for an ensemble of 1×10^6 trajectories. The curves are the fits to the quantum diffusion model distribution, including the effect of T_1, and with the evoltion parameter $\tau \equiv \bar{g}t \in [0, 1.2]$.



The best fit values of the integrated measurement coupling $\tau \equiv \int_0^t g\ dt$, when experimental data for weak Z-measurement of a superconducting transmon qubit with different initial states $\rho_{00}(0)$, are compared to the theoretical predictions. It is obvious that τ is independent of the initial state, and varies almost linearly with time after a slower initial build-up.

Fluctuation-Dissipation Relation

The geodesic parameter is $\rho_{00} - \rho_{11}$, with fixed points at ± 1.
The size of the fluctuations is, dropping the subleading $o(dt)$ terms:

$$\langle\langle (d\rho_{00} - d\rho_{11})^2 \rangle\rangle = 16 g^2 S_\xi \ \rho_{00}^2 \rho_{11}^2 \ dt \ .$$

The geodesic evolution term is:

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$$\langle\!\langle (d\rho_{00} - d\rho_{11})^2 \rangle\!\rangle = 4\rho_{00}\rho_{11} \frac{(d\rho_{00} - d\rho_{11})_{\rm geo}}{\rho_{00} - \rho_{11}} \ .$$

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In general stochastic processes, vanishing drift and fluctuation-dissipation relation are quite unrelated properties, involving first and second moments of the distribution respectively. The fact that both lead to the Born rule is an exceptional feature of quantum trajectory dynamics.

Implication: The environment can influence the measurement process only via the apparatus.

Notable Features

• Individual quantum trajectories evolve unitarily, even in presence of the noise. Mixed states arise when multiple trajectories with different noise histories are averaged over.

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• Measurement outcomes are independent of $\rho_{i\neq j}$, and so are unaffected by decoherence. A different noise can be added to the phases of $\rho_{i\neq j}$ without spoiling the evolution of ρ_{ii} and conflicting with the Born rule.

Origin of Noise

The quadratically nonlinear quantum measurement equation for state collapse supplements the Schrödinger evolution:

$$d\rho = i[\rho, H]dt + \sum_i w_i \; g[\rho P_i + P_i \rho - 2\rho Tr(\rho P_i)] \; dt + noise \; .$$

The underlying dynamics is the system-apparatus measurement interaction, and the nature of the noise depends on it.

What mechanism can simultaneously produce attraction towards the measurement eigenstates (geodesic evolution) and irreducible noise (stochastic fluctuations), with precisely related magnitudes?

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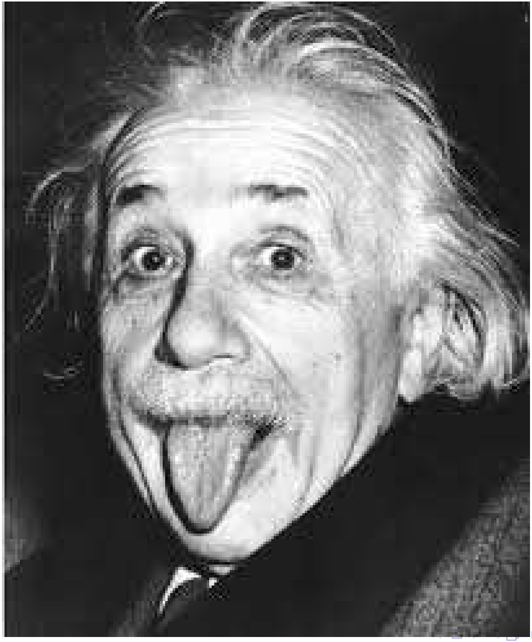
The measurement problem, i.e. the location of the "Heisenberg Cut" separating the quantum and the classical behaviour, is thus shifted higher up in the dynamics of the apparatus-dependent amplification.

Einstein strikes back!



Work in Progress

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Coherent states that continuously interpolate between quantum and classical regimes are a convenient choice for the apparatus pointer states.

$$|\alpha\rangle \equiv e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ .$$

Coherent states are the minimum uncertainty (equal to the zero-point fluctuations) states in the Fock space.

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The von Neumann interaction can amplify α and separate the pointer states. For measurement of a qubit using the electromagnetic field in a cavity, the von Neumann interaction gives:

$$H_{\rm int} = ig \ |1\rangle\langle 1| \otimes (a^\dagger - a) \ ,$$

$$|0\rangle_S |0\rangle_A \longrightarrow |0\rangle_S |0\rangle_A \ , \quad |1\rangle_S |0\rangle_A \longrightarrow |1\rangle_S |\alpha = gt\rangle_A \ .$$

Irreversibility needs to be added to this dynamics, possibly as a boundary condition, to convert entanglement into measurement.

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